Midterm Exam

Electricity and Magnetism 1

Friday May 23, 2014
09:00-11:00

Place your student card at the right side of the table.

Write your name and student number on *every* sheet.
***Write clearly*.**
Use a *separate* sheet for each question.

This exam consists of 3 questions.
All questions are of equal weight.

PROBLEM 1

Score: a+b+c+d+e =4+7+7+6+6=30

A long solid cylinder of radius R_1 carries a volume charge density $\rho(s) = Cs$; $0 \leq s \leq R_1$, where C is a *positive* constant and s the distance to the symmetry-axis of the cylinder. The solid cylinder is surrounded (coaxially) by a infinitely-thin negatively charged cylindrical shell of radius $R_2 > R_1$. The shell carries a uniform surface charge density σ. The magnitude of this surface charge density is such that the electric field is zero for $s > R_2$. You may assume that the cylinders are infinitely long and consequently edge effects may be neglected.

a) Give the units of C and σ.

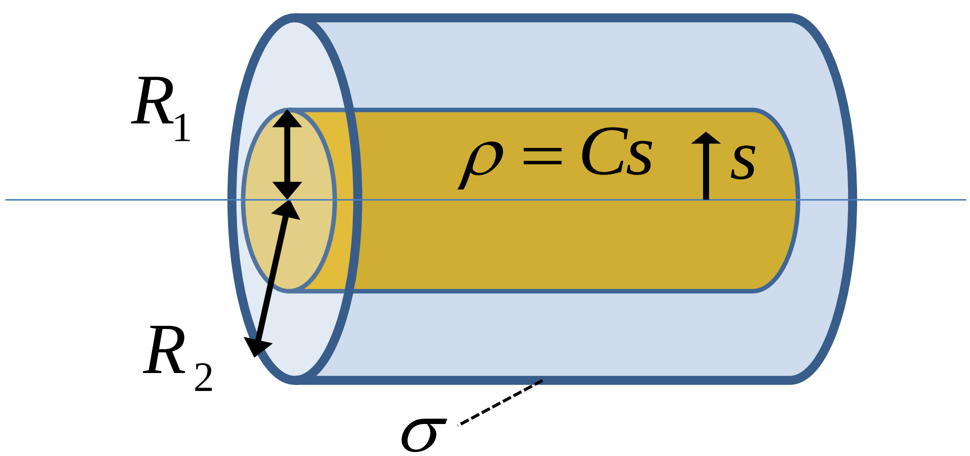
b) Show that σ can be expressed as:

$$\sigma = -C\frac{R_1^3}{3R_2}$$

c) Find the electric field $\vec{E}$ (magnitude and direction) in the region $0 \leq s \leq R_1$.

d) Find the electric field $\vec{E}$ (magnitude and direction) in the region $R_1 < s \leq R_2$.

e) A proton with charge $+e$ and mass m_p is released from rest at $s = R_1$. What is the speed v_p of the proton when it reaches $s = R_2$. Express your answer in terms of C, e, m_p, R_1, and R_2 as needed.



PROBLEM 2

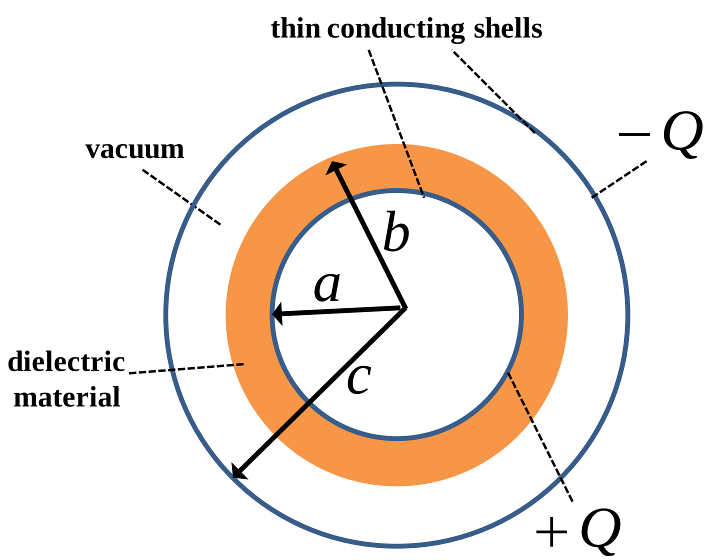
Score: a+b+c+d+e=6+6+6+6+6=30

A spherical capacitor consists of two (infinitesimally) thin conducting spherical shells. The inner conducting shell has radius a and the outer conducting shell has radius c. The space between the conductors is partly filled (from $r = a$ to $r = b$) with a linear dielectric material with electric susceptibility χ. The rest of the space between the conductors (from $r = b$ to $r = c$) is empty (vacuum). The inner conducting shell carries a charge $+Q$ and the outer conduction shell carries a charge $-Q$.

a) Find the electric displacement $\vec{D}$ (magnitude and direction) in the region $a < r < c$.
b) Find the electric field $\vec{E}$ (magnitude and direction) in the region $a < r < c$.
c) Find the potential difference ΔV between the inner conducting shell and the outer conducting shell.
d) Determine the capacitance of this system of two conducting shells.

We now connect the conducting surfaces of this spherical capacitor to a battery that keeps the potential difference between the conducting spheres at a constant value of V_{bat}. With the battery connected the *entire* region between the two shells is filled with the linear dielectric material with electric susceptibility χ.

e) By what amount ($\Delta Q = Q_{final} - Q_{initial}$) has the free charge on the positive spherical shell changed in going from the initially partly filled configuration to the final entirely filled configuration?

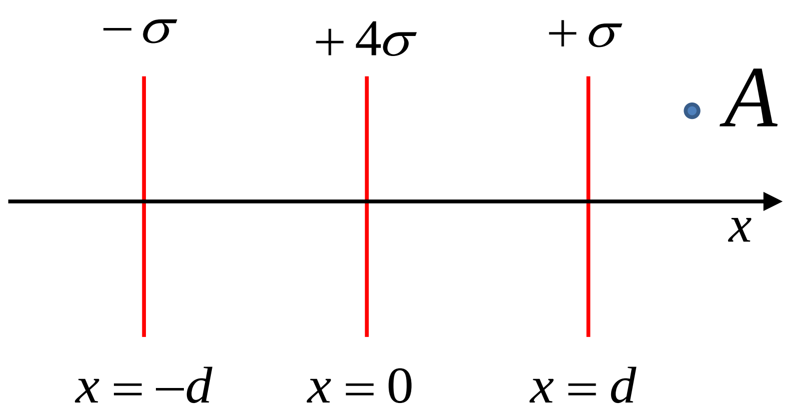


PROBLEM 3

Score: a+b+c+d+e =6+6+6+6+6=30

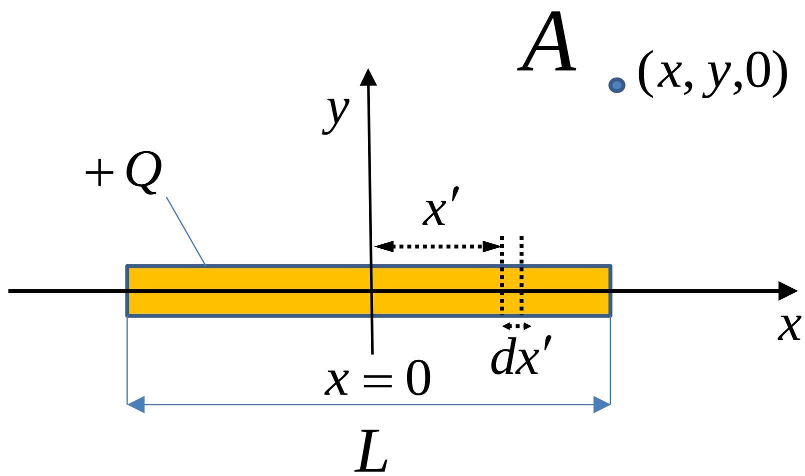
In this problem you are asked to answer 5 multiple choice questions. You do not need to explain your answer but you can simply write down the number of the correct answer.

a. Three infinite sheets of charge lying in the *yz*-plane are shown in the figure. The sheet on the right at $x = d$ is positively charged with charge per unit area of $+\sigma$, the sheet in the middle at $x = 0$ is positively charged with charge per unit area of $+4\sigma$, and the sheet on the left at $x = -d$ is negatively charged with charge per unit area of $-\sigma$. What is the electric field at the point A in the region $x > d$?



1. $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$
2. $\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{x}$
3. $\vec{E} = \frac{2\sigma}{\varepsilon_0}\hat{x}$
4. $\vec{E} = -\frac{2\sigma}{\varepsilon_0}\hat{x}$
5. $\vec{E} = \frac{3\sigma}{\varepsilon_0}\hat{x}$
6. $\vec{E} = -\frac{3\sigma}{\varepsilon_0}\hat{x}$
7. $\vec{E} = \frac{4\sigma}{\varepsilon_0}\hat{x}$
8. $\vec{E} = -\frac{4\sigma}{\varepsilon_0}\hat{x}$
9. Zero
10. None of the above

b. A thin rod extends along the x-axis from $x = -L/2$ to $x = L/2$. The rod carries a uniformly distributed positive charge $+Q$. Consider a point A in the $z = 0$ plane with coordinates $(x, y, 0)$. Which of the following expressions describes the electric potential difference $V(A) - V(\infty)$, between infinity and the point A?



1. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{(x-\acute{x})\hat{x}+y\hat{y}}{((x-\acute{x})^2+y^2)^{3/2}} d\acute{x}$

2. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{(x-\acute{x})\hat{x}+y\hat{y}}{((x-\acute{x})^2+y^2)^{1/2}} d\acute{x}$

3. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{(x-\acute{x})}{((x-\acute{x})^2+y^2)^{3/2}} d\acute{x}$

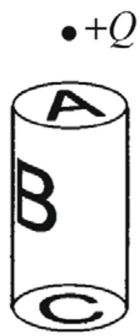
4. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{(x-\acute{x})}{((x-\acute{x})^2+y^2)^{1/2}} d\acute{x}$

5. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{1}{((x-\acute{x})^2+y^2)^{3/2}} d\acute{x}$

6. $V(A) - V(\infty) = \frac{Q}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2} \frac{1}{((x-\acute{x})^2+y^2)^{1/2}} d\acute{x}$

7. None of the above

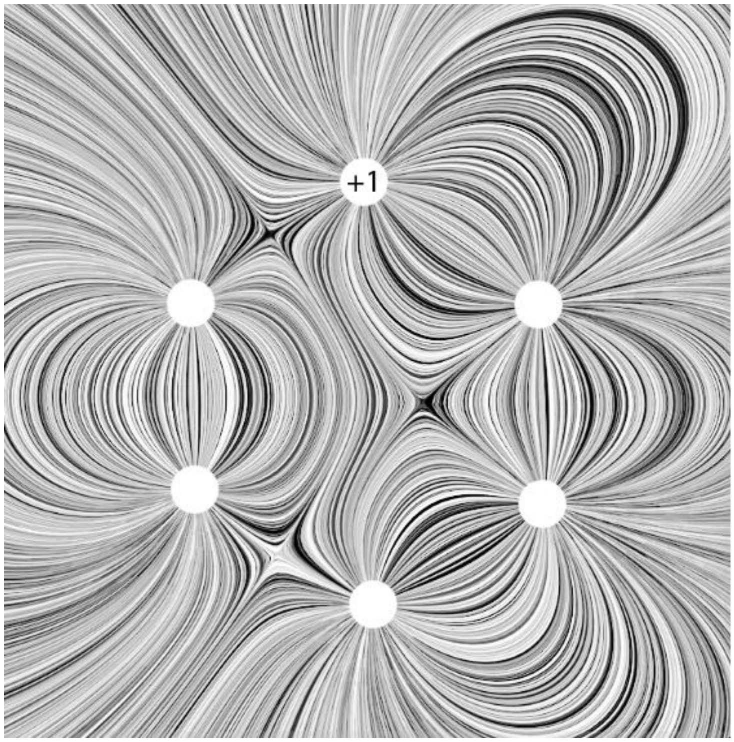
c. The area vector $d\vec{a}$ at each point on a closed surface (i.e., a surface that surrounds a volume) is always chosen to point out of the enclosed volume. Consider a closed imaginary surface (Gaussian surface) in the shape of a cylinder. A positive charge is located on the cylinder axis above the Gaussian cylinder, as shown in the figure below.



Which statement is correct about the flux $\Phi(B) = \int_B \vec{E} \cdot d\vec{a}$ through surface B and the flux $\Phi(A + B + C) = \oint_{A+B+C} \vec{E} \cdot d\vec{a}$ through the entire closed surface A+B+C ?

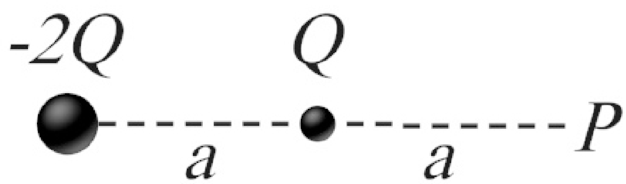
1. The flux $\Phi(B)$ is positive and the flux $\Phi(A + B + C)$ is positive.
2. The flux $\Phi(B)$ is positive and the flux $\Phi(A + B + C)$ is negative.
3. The flux $\Phi(B)$ is positive and the flux $\Phi(A + B + C)$ is zero.
4. The flux $\Phi(B)$ is negative and the flux $\Phi(A + B + C)$ is positive.
5. The flux $\Phi(B)$ is negative and the flux $\Phi(A + B + C)$ is negative.
6. The flux $\Phi(B)$ is negative and the flux $\Phi(A + B + C)$ is zero.
7. The flux $\Phi(B)$ is zero and the flux $\Phi(A + B + C)$ is positive.
8. The flux $\Phi(B)$ is zero and the flux $\Phi(A + B + C)$ is negative.
9. The flux $\Phi(B)$ is zero and the flux $\Phi(A + B + C)$ is zero.
10. Both fluxes are undefined.
11. None of the above.

d. Six charges all have a charge of either +1 C or -1 C. They are arranged as shown in the field lines representation of the field in the figure below. The top charge has a charge of +1 C. What is the total charge of the six charges?



1. +4 C
2. +3 C
3. +2 C
4. +1 C
5. 0 C
6. -1 C
7. -2 C
8. -3 C
9. -4 C

e. An observer sits at point *P* a distance *a* away from a charge *Q* and a distance *2a* away from a charge *-2Q*. The two charges and the observer all lie on the same line, as shown in the figure below. If we take the electric potential to be zero at infinity, then which of the following statements is true about the electric potential and electric field at point *P*? Assume *Q* > 0.



1. The electric potential at *P* is less than zero and the electric field at *P* is equal to zero.
2. The electric potential at *P* is less than zero and the electric field at *P* points to the right.
3. The electric potential at *P* is less than zero and the electric field at *P* points to the left.
4. The electric potential at *P* is equal to zero and the electric field at *P* is equal to zero.
5. The electric potential at *P* is equal to zero and the electric field at *P* points to the right.
6. The electric potential at *P* is equal to zero and the electric field at *P* points to the left.
7. The electric potential at *P* is greater than zero and the electric field at *P* is equal to zero.
8. The electric potential at *P* is greater than zero and the electric field at *P* points to the right.
9. The electric potential at *P* is greater than zero and the electric field at *P* points to the left.

Solutions

PROBLEM 1

a)

The units of C are Coulomb·m^{-4} (charge per volume from the charge density and then divided by the length unit of the coordinate s.

The units of a surface charge density are Coulomb·m^{-2}.

b)

Use Gauss's law (cylinder symmetry; fields are in the *s*-direction) with a cylindrical surface of length L and a radius $s > R_2$. Then,

$$\oint \vec{E} \cdot d\vec{a} = 2\pi sLE_{outside} = 0 = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(2\pi R_2 L\sigma + \int_0^{R_1} 2\pi \acute{s}\, L\rho(\acute{s})d\acute{s}\right) \Rightarrow$$

$$0 = \frac{1}{\varepsilon_0}\left(2\pi R_2 L\sigma + \int_0^{R_1} 2\pi \acute{s}^2\, LCd\acute{s}\right) = \frac{1}{\varepsilon_0}\left(2\pi R_2 L\sigma + \frac{2}{3}\pi R_1^3 LC\right) \Rightarrow$$

$$\sigma = -C\frac{R_1^3}{3R_2}$$

c)

Use Gauss's law:

$$\oint \vec{E} \cdot d\vec{a} = 2\pi sLE = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_0^{s} 2\pi \acute{s}\, L\rho(\acute{s})d\acute{s} \Rightarrow$$

$$E = \frac{C}{\varepsilon_0 s}\int_0^{s} \acute{s}^2\, d\acute{s} = \frac{Cs^2}{3\varepsilon_0} \Rightarrow \vec{E} = \frac{Cs^2}{3\varepsilon_0}\,\hat{s}$$

d)

Use Gauss's law:

$$\oint \vec{E} \cdot d\vec{a} = 2\pi sLE = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_0^{R_1} 2\pi \acute{s}\, L\rho(\acute{s})d\acute{s} \Rightarrow$$

$$E = \frac{C}{\varepsilon_0 s}\int_0^{R_1} \acute{s}^2\, d\acute{s} = \frac{CR_1^3}{3\varepsilon_0 s} \Rightarrow \vec{E} = \frac{CR_1^3}{3\varepsilon_0 s}\,\hat{s}$$

e)

Work W required to move the proton from $s = R_1$ to $s = R_2$ is:

$$W = e(V(s = R_2) - V(s = R_1)) = -e\int_{R_1}^{R_2} \vec{E}\cdot d\vec{s} = -e\int_{R_1}^{R_2} \frac{CR_1^3}{3\varepsilon_0 s}\,\hat{s}\cdot d\vec{s} \Rightarrow$$

$$W = -e\int_{R_1}^{R_2} \frac{CR_1^3}{3\varepsilon_0 s} ds = e\frac{CR_1^3}{3\varepsilon_0}\ln\frac{R_1}{R_2} < 0$$

This is negative amount of energy and the positive amount is gained by the proton and converted into kinetic energy:

$$\frac{1}{2}m_p v_p^2 = e\frac{CR_1^3}{3\varepsilon_0}\ln\frac{R_2}{R_1} \Rightarrow v_p = \sqrt{\frac{2e}{m_p}\frac{CR_1^3}{3\varepsilon_0}\ln\frac{R_2}{R_1}}$$

PROBLEM 2:

a)

Use Gauss's law for the electric displacement $\vec{D}$ and considering a spherical shell with radius $a < r < c$.

$$\oint \vec{D} \cdot d\vec{a} = Q_{enc}^{free} = Q = 4\pi r^2 D \Rightarrow$$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

b)

Use

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\vec{D}}{\varepsilon_0(1+\chi)}$$

In the region $a < r < b$, with the linear dielectric material:

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0(1+\chi)} = \frac{Q}{4\pi\varepsilon_0(1+\chi)r^2}\hat{r}$$

In the region $b < r < c$, we have $\chi = 0$ (vacuum):

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0(1+\chi)} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

c)

$$\Delta V = V_+ - V_- = -\int_{-}^{+} \vec{E} \cdot d\vec{r} = -\int_{c}^{a} \vec{E} \cdot d\vec{r}$$

$$= -\left(\int_{c}^{b} \frac{Q}{4\pi\varepsilon_0 r^2} dr + \int_{b}^{a} \frac{Q}{4\pi\varepsilon_0(1+\chi)r^2} dr\right) \Rightarrow$$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{Q}{4\pi\varepsilon_0(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\left[\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

d)

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left[\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)\right]}$$

e)
Use
$Q = C$V.

$$Q_{initial} = C_{initial}V_{bat} = \frac{4\pi\varepsilon_0 V_{bat}}{\left[\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)\right]}$$

$$Q_{final} = C_{final}V_{bat} = \frac{4\pi\varepsilon_0 V_{bat}}{\left[\frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{c}\right)\right]}$$

$$\Delta Q = \frac{4\pi\varepsilon_0 V_{bat}}{\left[\frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{c}\right)\right]} - \frac{4\pi\varepsilon_0 V_{bat}}{\left[\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)\right]}$$

PROBLEM 3

a) 3
b) 6
c) 3
d) 5
e) 5